

UNIT - 5

①

PRINCIPAL STRESSES & STRAINS

[Ref: Strength of material - Dr. R.K. Bansal]

Principal planes:-

It has been observed that at any point in a strained material, there are three planes, mutually perpendicular to each other, which carry direct stresses only, and no shear stress. A little consideration will show that out of these direct stresses one will be maximum, the other minimum and the third an intermediate between the two. These particular planes which have no shear stress, are known as principal planes.

Principal stress:-

The magnitude of direct stress, across a principal plane is known as principal stress. The determination of principal planes, and then principle stress is an important factor in the design of various structures and machine components.

Methods for the stresses on an oblique section of a body:

The following two methods for the

determination of stresses on an oblique section.

1. Analytical method
2. Graphical method.

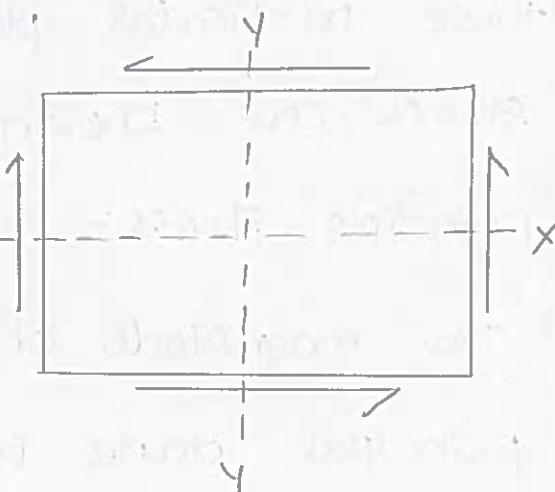
Analytical Method for the stress on an oblique section of a body:-

The following cases will be kept in mind while for the determination of stresses on an oblique section.

1. A body subjected to a direct stress in one plane.
2. A body subjected to direct stresses in two mutually perpendicular directions.

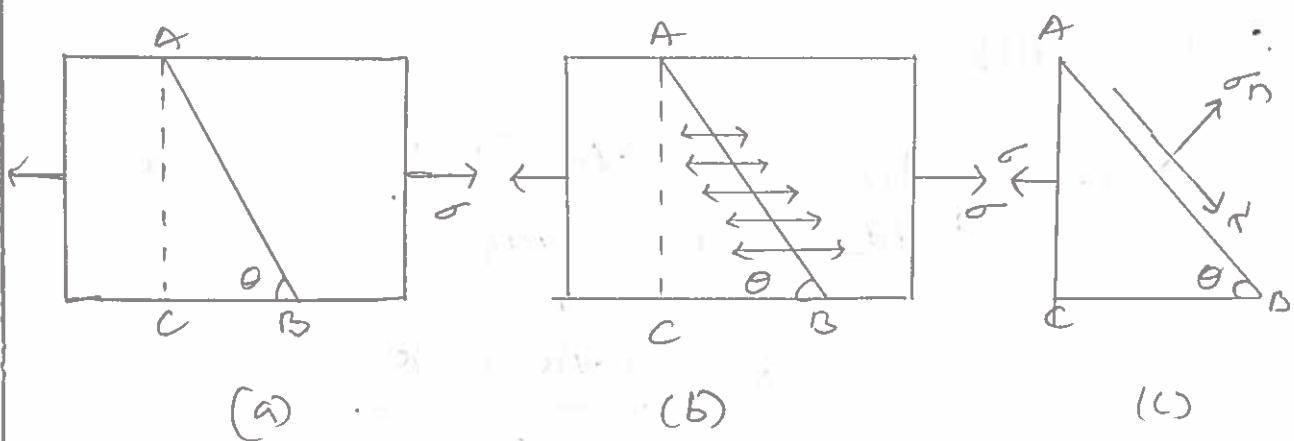
Sign Conventions for Analytical Method:

1. All the tensile stresses and strains are taken as positive, whereas all the compressive stresses and strains are taken as negative.



2. The shear stress which tends to rotate the element in the clockwise direction is taken as positive, whereas that which tends to rotate in an anticlockwise direction as negative.

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 Stresses on an oblique section of a body subjected to a direct stress in one plane:
 consider a rectangular body of uniform cross-sectional area and const thickness subjected to a direct tensile stress along $x-x$ axis as shown in fig. Now let us consider an oblique section AB inclined with the $x-x$ axis. with the line of action of the tensile stress which we are required to find out the stresses as shown in fig.



let τ = tensile stress across the face AC
and

θ = angle which the oblique section AB makes with BC i.e with the $x-x$ axis in the clockwise direction.

Consider the equilibrium of an element or wedge ABC whose free body diagram is shown in fig (b) & (c). we know that the horizontal force acting on the face AC

$$P = \sigma \cdot AC (\leftarrow)$$

Resolving the force perpendicular or normal to the section AB.

$$\begin{aligned} P_n &= P \sin \theta \\ &= \sigma \cdot AC \sin \theta \quad \text{--- (i)} \end{aligned}$$

And now resolving the force tangential to the Section AD

$$\begin{aligned} P_t &= P \cos \theta \\ &= \sigma \cdot AC \cos \theta \quad \text{--- (ii)} \end{aligned}$$

we know that normal stress across the section AB.

$$\begin{aligned} \sigma_n &= \frac{P_n}{AB} = \frac{\sigma \cdot AC \sin \theta}{AB} \\ &= \frac{\sigma \cdot AC \sin \theta}{\frac{AC}{\sin \theta}} \\ &= \sigma \sin^2 \theta \\ &= \frac{\sigma}{2} (1 - \cos 2\theta) \\ &= \frac{\sigma}{2} - \frac{\sigma}{2} \cos 2\theta \quad \text{--- (3)} \end{aligned}$$

and shear stress across the Section AB.

$$\tau = \frac{P_t}{AB} = \frac{\sigma \cdot AC \cdot \cos \theta}{AB}$$

(3)

$$= \frac{\sigma \cdot AC \cos\theta}{\sin\theta}$$

$$= \sigma \cdot \sin\theta \cdot \cos\theta$$

$$= \frac{\sigma}{2} \sin 2\theta \quad \text{--- (4)}$$

from the above equations the shear stress will be maximum when $\sin\theta = 1$ or $\sin\theta = 1$ or $\theta = 90^\circ$, or in other words, the face AC will carry the maximum direct stress similarly, the shear stress across the section AB will be maximum when $\sin 2\theta = 1$ or $2\theta = 90^\circ$ or 270° , or in other words the shear stress will be maximum on the plane inclined at the 125° with the line of action of the tensile stress.

\therefore maximum shear stress when θ is equal to 45° .

$$\tau_{\max} = \frac{\sigma}{2} \sin 90^\circ$$

$$= \frac{\sigma}{2} \times 1$$

$$= \frac{\sigma}{2}$$

and maximum shear stress when θ is equal to 45° & 135°

$$\begin{aligned} \tau_{\max} &= \frac{\sigma}{2} \sin 90^\circ \\ &= \frac{\sigma}{2} \times 1 \\ &= \frac{\sigma}{2} \end{aligned}$$

and maximum shear stress when θ is equal to 45° or 135°

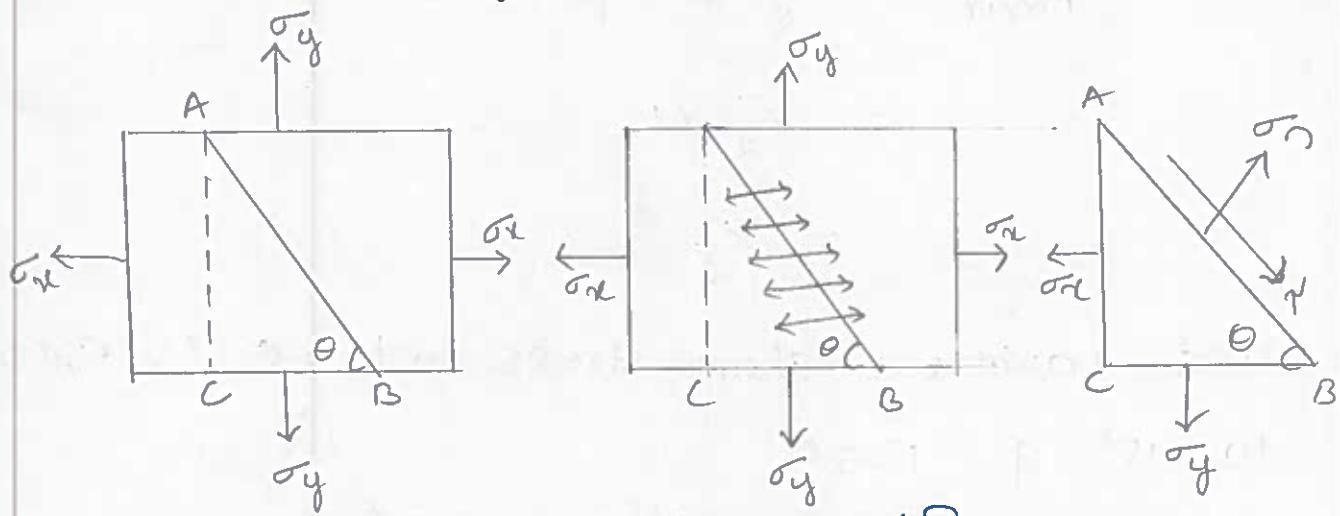
$$\begin{aligned} \tau_{\max} &= -\frac{\sigma}{2} \sin (270^\circ) \\ &= -\frac{\sigma}{2} (-1) \end{aligned}$$

$$\tau_{\max} = \frac{\sigma}{2}$$

It is thus obvious that the magnitudes of maximum shear stress is half of the tensile stress. Now the resultant stress may be found out from the relation.

$$\sigma_r = \sqrt{\sigma_r^2 + \tau^2}$$

Q: Stresses on an oblique section of a body subjected to direct stresses in two mutually perpendicular directions :-



(+) Consider a rectangular body of uniform cross-sectional and unit thickness subjected to direct tensile stresses in two mutually perpendicular directions along $x-x$ and $y-y$ areas as shown in fig. Now let us consider an oblique section AB inclined with $x-x$ axis.

Let σ_x = tensile stress along $x-x$ axis
(major tensile stress)

σ_y = tensile stress along $y-y$ axis
(minor tensile stress)

θ = angle which the oblique section AB makes with $x-x$ axis.

Consider the equilibrium of the wedge ABC. We know that horizontal force acting on the face AC (or $x-x$ axis)

$$P_x = \sigma_x AC (\leftarrow)$$

and vertical force acting on the face BC (or $y-y$ axis)

$$P_y = \sigma_y BC (\downarrow)$$

Resolving the forces perpendicular or normal to the section AB

$$P_n = P_x \sin \theta + P_y \cos \theta$$

$$P_n = \sigma_x AC \sin \theta + \sigma_y BC \cos \theta \quad \text{--- (1)}$$

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and now resolving the forces tangential to the section AB.

$$P_t = P_x \cos\theta - P_y \sin\theta$$

$$= \sigma_x AC \cos\theta - \sigma_y BC \sin\theta$$

$$P_t = \sigma_x AC \cos\theta - \sigma_y BC \sin\theta - \textcircled{2}$$

we know that normal stress across the section AB,

$$\sigma_n = \frac{P_n}{AB} = \frac{\sigma_x AC \sin\theta + \sigma_y BC \cos\theta}{AB}$$

$$= \frac{\sigma_x AC \sin\theta}{AB} + \frac{\sigma_y BC \cos\theta}{AB}$$

$$= \frac{\sigma_x AC \cdot \sin\theta}{AC} + \frac{\sigma_y BC \cdot \cos\theta}{BC \cos\theta}$$

$$= \sigma_x \sin^2\theta + \sigma_y \cos^2\theta$$

$$= \frac{\sigma_x}{2} (1 - \cos 2\theta) + \frac{\sigma_y}{2} (1 + \cos 2\theta)$$

$$= \frac{\sigma_x}{2} - \frac{\sigma_x}{2} \cos 2\theta + \frac{\sigma_y}{2} + \frac{\sigma_y}{2} \cos 2\theta$$

$$\boxed{\sigma_n = \frac{\sigma_x + \sigma_y}{2} - \left[\frac{\sigma_x - \sigma_y}{2} \right] \cos 2\theta} - \textcircled{3}$$

and shear stress across the section AB

$$\tau = \frac{P_t}{AB} = \frac{\sigma_x AC \cos\theta - \sigma_y BC \sin\theta}{AB}$$

$$\begin{aligned}
 &= \frac{\sigma_x AC \sin\theta}{AB} + \frac{\sigma_y BC \cos\theta}{AB} \\
 &= \frac{\sigma_x AC \sin\theta}{\frac{AC}{\sin\theta}} + \frac{\sigma_y BC \cos\theta}{\frac{BC}{\cos\theta}} \\
 &= \sigma_x \sin^2\theta + \sigma_y \cos^2\theta \\
 &= \frac{\sigma_x}{2} (1 - \cos 2\theta) + \frac{\sigma_y}{2} (1 + \cos 2\theta) \\
 &= \frac{\sigma_x}{2} - \frac{\sigma_x}{2} \cos 2\theta + \frac{\sigma_y}{2} + \frac{\sigma_y}{2} \cos 2\theta
 \end{aligned}$$

$$\sigma_n = \frac{\sigma_x + \sigma_y}{2} - \left[\frac{\sigma_x - \sigma_y}{2} \right] \cos 2\theta \quad \text{--- (6)}$$

and shear stress across the section AB

$$\tau = \frac{Pt}{AB} = \frac{\sigma_x AC \cos\theta - \sigma_y BC \sin\theta}{AB}$$

$$= \frac{\sigma_x AC \cos\theta}{AB} - \frac{\sigma_y BC \sin\theta}{AB}$$

$$= \frac{\sigma_x AC \cos\theta}{\frac{AC}{\sin\theta}} - \frac{\sigma_y BC \sin\theta}{\frac{BC}{\cos\theta}}$$

$$= \sigma_x \sin\theta \cos\theta - \sigma_y \sin\theta \cos\theta$$

$$= (\sigma_x - \sigma_y) \sin\theta \cos\theta$$

$$\boxed{\tau = \frac{\sigma_x - \sigma_y}{2} \sin 2\theta}$$

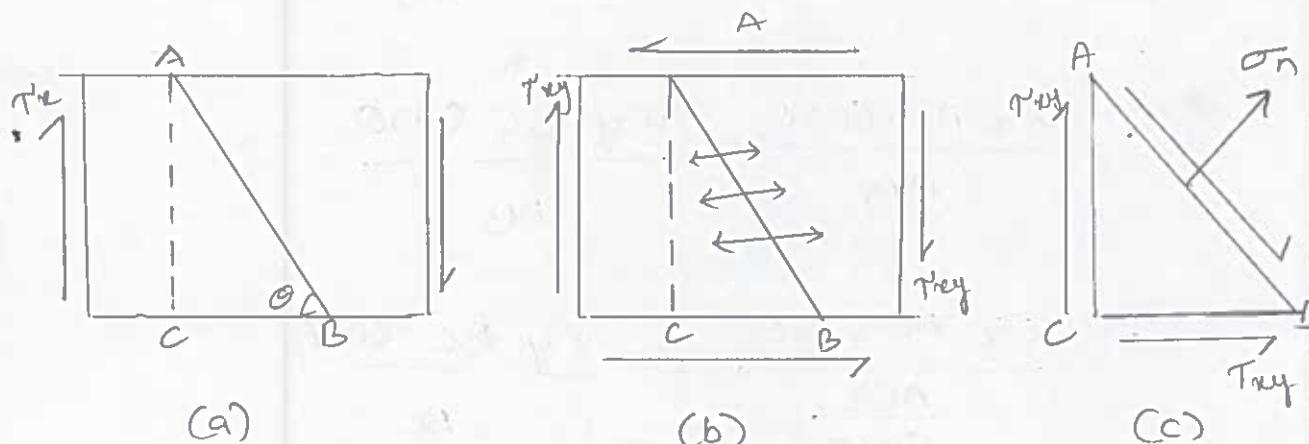
It will be interesting to know from equation (3) shear stress across the section AB will be maximum when $\sin 2\theta = 1$ or $2\theta = 90^\circ$ (or) $\theta = 45^\circ$; therefore maximum shear stress.

$$\tau_{\max} = \frac{\sigma_x - \sigma_y}{2}$$

Now the resultant stress may be found out from the relation.

$$\sigma_R = \sqrt{\sigma_n^2 + \tau^2}$$

3) Stresses on An oblique section of A Body Subjected To A simple shear stress-



Consider a rectangular body of uniform cross-sectional area and unit thickness subjected to a positive Cise (clock wise) shear stress along x-x axis as shown in fig (a) Now let us consider an oblique

section AB inclined with x-x axis on which we are required to find out the stresses as shown in fig (b)

let τ_{xy} = positive shear stress along x-x axis.

θ = angle which the oblique section AB makes with x-x axis in the anti clockwise direction.

Consider the equilibrium of the wedge ABC. We know that as per the principle of simple shear, the face BC, of the wedge will be subjected to an anticlockwise shear stress equal to τ_{xy} as shown in fig.(B).

We know that vertical force acting on the face AC,

$$P_1 = \tau_{xy} AC (\uparrow)$$

and horizontal force acting on the face BC,

$$P_2 = \tau_{xy} BC (\rightarrow)$$

Resolving the forces perpendicular or normal to the AB,

$$P_n = P_1 \cos\theta + P_2 \sin\theta$$

$$= \tau_{xy} AC \cdot \cos\theta + \tau_{xy} BC \sin\theta$$

and now revolving the forces tangential to the section AB

$$P_t = P_2 \sin\theta - P_1 \cos\theta$$

$$= T_{xy} BC \sin\theta - T_{xy} AC \cos\theta$$

we know that normal stress across the section AB.

$$\sigma_n = \frac{P_n}{AB} = \frac{T_{xy} AC \cos\theta + T_{xy} BC \sin\theta}{AB}$$

$$= \frac{T_{xy} AC \cos\theta}{AB} + \frac{T_{xy} BC \sin\theta}{\frac{BC}{\cos\theta}}$$

$$= T_{xy} \sin\theta \cos\theta + T_{xy} \sin\theta \cos\theta$$

$$= 2 T_{xy} \sin\theta \cos\theta = T_{xy} \sin 2\theta$$

and shear stress across the section AB

$$\tau = \frac{P_t}{AB} = \frac{T_{xy} BC \sin\theta - T_{xy} AC \cos\theta}{AB}$$

$$= \frac{T_{xy} BC \sin\theta}{AB} - \frac{T_{xy} AC \cos\theta}{AB}$$

$$= \frac{T_{xy} BC \sin\theta}{\frac{BC}{\sin\theta}} - \frac{T_{xy} AC \cos\theta}{\frac{AC}{\cos\theta}}$$

$$= T_{xy} \sin^2\theta - T_{xy} \cos^2\theta$$

$$= \frac{\tau_{xy}}{2} (1 - \cos 2\theta) - \frac{\tau_{xy}}{2} (1 + \cos 2\theta)$$

$$= \frac{\tau_{xy}}{2} - \frac{\tau_{xy}}{2} \cos 2\theta - \frac{\tau_{xy}}{2} - \frac{\tau_{xy}}{2} \cos 2\theta$$

$$= - \frac{\tau_{xy}}{2} \cos 2\theta$$

(minus sign means that normal stress is opposite to that across AC.)

Now the planes of maximum and minimum normal stresses may be found out by equating the shear stress to zero

$$\text{i.e. } \sigma_{xy} \cos 2\theta = 0$$

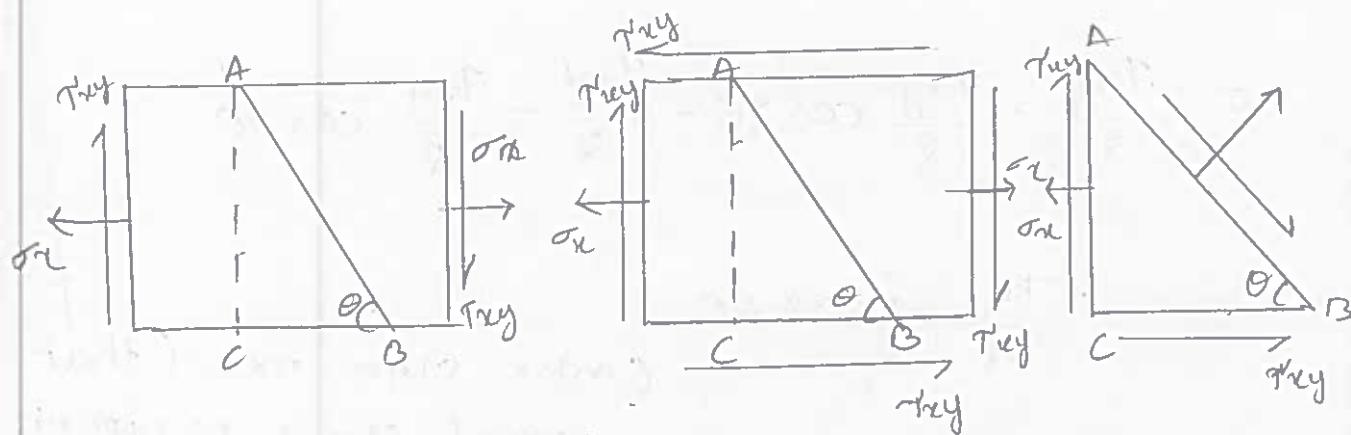
The above equation is possible only if

$$3\theta = 90^\circ \text{ or } 270^\circ \text{ (because } \cos 90^\circ \text{ or } \cos 270^\circ = 0) \text{ or in other words } \theta = 45^\circ \text{ or } 135^\circ$$

Stresses on An oblique Section of A body of
A Body Subjected to A Direct in one
plane And Accompanied By a simple shear
stress:

Consider a rectangular body of uniform
cross-sectional area and unit thickness
subjected to a tensile stress along x-x
axis accompanied by a positive (ie clock
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Shear stresses along $x-x$ axis as shown in fig.



Now let us consider an oblique section AB inclined with $x-x$ axis on which we are required to find out the stresses as shown in fig.

Let

σ_x = Tensile stresses along $x-x$ axis

τ_{xy} = positive shear stresses along $x-x$ axis, and

θ = angle which the oblique section AB makes with $x-x$ axis in clockwise direction.

Consider the equilibrium of the wedge ABC, we know that as per the principle of simple shear, the face BC of the wedge will be subjected to an anticlockwise shear stress equal to τ_{xy} . As shown fig(c) we know that horizontal force acting on the face AC.

$$P_x = \sigma_x AC (\leftarrow) \quad \text{--- (1)}$$

similarly vertical force acting on the face AC

$$P_y = T_{xy} AC (\uparrow) \quad \text{--- (2)}$$

and horizontal force acting on the face BC,

$$P = T_{xy} BC (\rightarrow) \quad \text{--- (3)}$$

Resolving the forces perpendicular to the section AB.

$$P_n = P_x \sin\theta - P_y \cos\theta - P \sin\theta$$

$$= \sigma_x AC \sin\theta - T_{xy} AC \cos\theta - T_{xy} BC \sin\theta$$

we know that normal stress across the section

AB,

$$\sigma_n = \frac{P_n}{AB} = \frac{\sigma_x AC \sin\theta - T_{xy} AC \cos\theta - T_{xy} BC \sin\theta}{AB}$$

$$= \frac{\sigma_x AC \sin\theta}{AB} - \frac{T_{xy} AC \cos\theta}{AB} - \frac{T_{xy} BC \sin\theta}{AB}$$

$$= \frac{\sigma_x AC \sin\theta}{\frac{AC}{\sin\theta}} - \frac{T_{xy} AC \cos\theta}{\frac{AC}{\sin\theta}} - \frac{T_{xy} BC \sin\theta}{\frac{BC}{\cos\theta}}$$

$$= \sigma_x \sin^2\theta - T_{xy} \sin\theta \cos\theta - T_{xy} \sin\theta \cos\theta$$

$$= \frac{\sigma_x}{2} (1 - \cos 2\theta) - 2 T_{xy} \sin\theta \cos\theta$$

$$= \frac{\sigma_x}{2} - \frac{\sigma_x}{2} \cos 2\theta - T_{xy} \sin 2\theta \quad \text{--- (4)}$$

Shear stress (i.e. tangential stress) across the section AB.

$$\tau = \frac{P_t}{AB} = \frac{\sigma_x AC \cos\theta + T_{xy} AC \sin\theta - T_{xy} BC \cos\theta}{AB}$$

$$= \frac{\sigma_x AC \cos\theta}{AB} + \frac{T_{xy} AC \sin\theta}{AB} - \frac{T_{xy} BC \cos\theta}{AB}$$

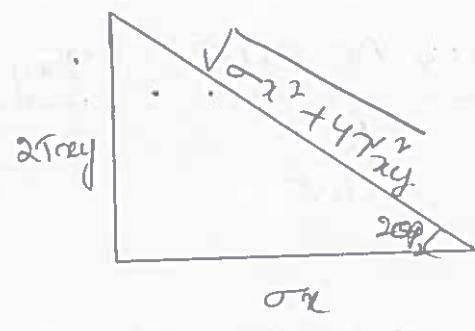
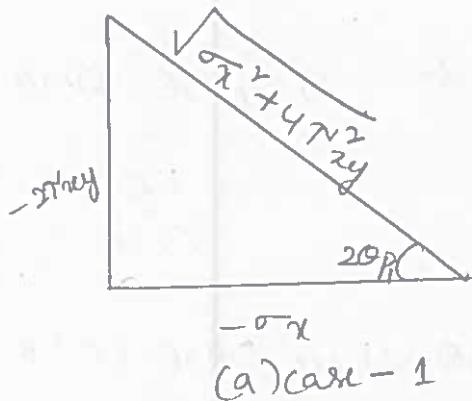
$$= \sigma_x \sin\theta \cos\theta + T_{xy} \sin^2\theta - T_{xy} \cos^2\theta$$

$$= \frac{\sigma_x}{2} (\sin 2\theta) + \frac{T_{xy}}{2} (1 - \cos 2\theta) - \frac{T_{xy}}{2} (1 + \cos 2\theta)$$

$$= \frac{\sigma_x}{2} \sin 2\theta + \frac{T_{xy}}{2} - \frac{T_{xy}}{2} \cos 2\theta - \frac{T_{xy}}{2} - \frac{T_{xy}}{2} \cos 2\theta$$

$$\tau = \frac{\sigma_x}{2} \sin 2\theta - T_{xy} \cos 2\theta \quad \text{--- (5)}$$

Now from the above equation we find that the following two cases satisfy this condition as shown in fig (a) & (b).



Thus we find that these are two principal planes at right angle to each other, their inclination with x-x axis being θ_p and θ_{p_2} .

(a)

$$\sin 2\theta_{P_1} = \frac{-2\tau_{xy}}{\sqrt{\sigma_x^r + 4\tau_{xy}^r}} \quad \text{and} \quad \cos 2\theta_{P_1} = \frac{-\sigma_x^r}{\sqrt{\sigma_x^r + 4\tau_{xy}^r}}$$

Similarly for case 2,

$$\sin 2\theta_{P_2} = \frac{2\tau_{xy}}{\sqrt{\sigma_x^r + 4\tau_{xy}^r}} \quad \text{and} \quad \cos 2\theta_{P_2} = \frac{\sigma_x^r}{\sqrt{\sigma_x^r + 4\tau_{xy}^r}}$$

Now the values of principal stresses may be found out by substituting the above values $2\theta_{P_1}$ and $2\theta_{P_2}$ in $\sin 2\theta$

$$\begin{aligned} &= \frac{\sigma_x}{2} - \frac{\sigma_x^r}{2} \times \frac{-\sigma_x^r}{\sqrt{\sigma_x^r + 4\tau_{xy}^r}} - \tau_{xy} \frac{-2\tau_{xy}}{\sqrt{\sigma_x^r + 4\tau_{xy}^r}} \\ &= \frac{\sigma_x}{2} + \frac{\sigma_x^r}{2\sqrt{\sigma_x^r + 4\tau_{xy}^r}} + \frac{2\tau_{xy}^r}{\sqrt{\sigma_x^r + 4\tau_{xy}^r}} \\ &= \frac{\sigma_x}{2} + \frac{\sigma_x^r + 4\tau_{xy}^r}{2\sqrt{\sigma_x^r + 4\tau_{xy}^r}} \\ &= \frac{\sigma_x}{2} + \frac{\sqrt{\sigma_x^r + 4\tau_{xy}^r}}{2} \end{aligned}$$

$$\boxed{\sigma_{P_1} = \frac{\sigma_x}{2} + \sqrt{\left(\frac{\sigma_x}{2}\right)^2 + \tau_{xy}^r}}$$

maximum principal stresses

$$\sigma_{P_2} = \frac{\sigma_x}{2} - \frac{\sigma_x}{2} \cos 2\theta - \tau_{xy} \sin 2\theta$$

$$= \frac{\sigma_x}{2} - \frac{\sigma_x}{2} \times \frac{\sigma_x}{\sqrt{\sigma_x^2 + 4\tau_{xy}^2}} - \tau_{xy} \frac{2\tau_{xy}}{\sqrt{\sigma_x^2 + 4\tau_{xy}^2}}$$

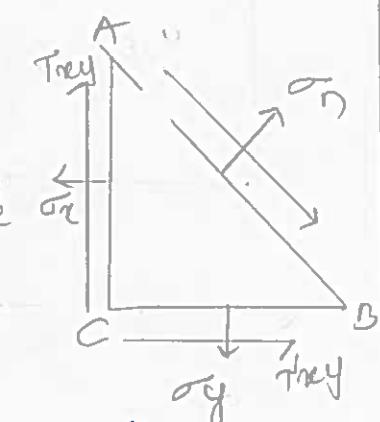
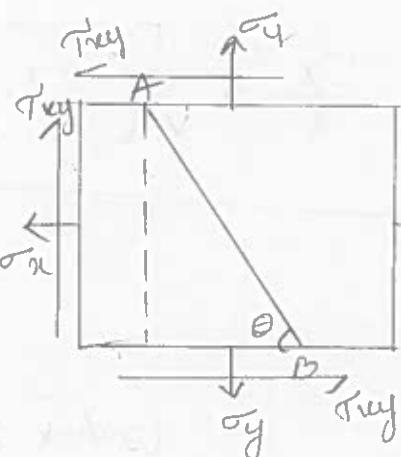
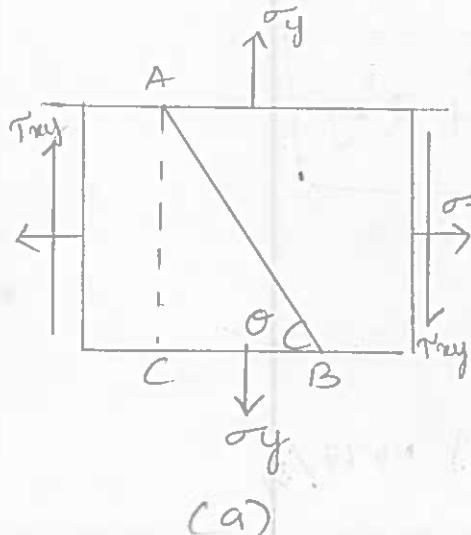
$$= \frac{\sigma_x}{2} - \frac{\sigma_x}{2\sqrt{\sigma_x^2 + 4\tau_{xy}^2}} - \frac{2\tau_{xy}}{\sqrt{\sigma_x^2 + 4\tau_{xy}^2}}$$

$$= \frac{\sigma_x}{2} - \frac{\sigma_x^2 + 4\tau_{xy}^2}{2\sqrt{\sigma_x^2 + 4\tau_{xy}^2}}$$

$$= \frac{\sigma_x}{2} - \frac{\sqrt{\sigma_x^2 + 4\tau_{xy}^2}}{2}$$

$$\sigma_{P_2} = \frac{\sigma_x}{2} - \sqrt{\left(\frac{\sigma_x}{2}\right)^2 + \tau_{xy}^2}$$

5. Stresses on An oblique Section of A Body
Subjected To Direct Stresses In Two
Mutually perpendicular Directions Accompanied
By A Simple Shear Stress:



(10)

consider a rectangular body of uniform cross-sectional area and unit thickness subjected to tensile stresses along x-x and y-y axes, and accompanied by a positive shear stress along x-x axis as shown in fig.

Now let us consider an oblique section AB inclined with x-x axis on which we are required to find out the stresses as shown in fig.

let σ_x = tensile stress along x-x axis

σ_y = tensile stress along y-y axis

τ_{xy} = positive shear stress along x-x axis

θ = angle of inclination.

consider the equilibrium of the wedge ABC. we know that the horizontal force acting on the face AC,

$$P_1 = \sigma_x AC (\leftarrow) \quad \text{---(1)}$$

And vertical force acting on the face AC

$$P_2 = \tau_{xy} AC (\uparrow) \quad \text{---(2)}$$

similarly, vertical force Acting on the face BC.

$$P_3 = \sigma_y BC (\downarrow) \quad \text{---(3)}$$

$$P_y = \tau_{xy} BC (\rightarrow) - \textcircled{4}$$

Now resolving the forces perpendicular to the section AB

$$\begin{aligned} P_n &= P_1 \sin\theta - P_2 \cos\theta + P_3 \cos\theta - P_4 \sin\theta \\ &= \sigma_x AC \sin\theta - \tau_{xy} AC \cos\theta + \sigma_y BC \cos\theta \\ &\quad - \tau_{xy} BC \sin\theta. \end{aligned}$$

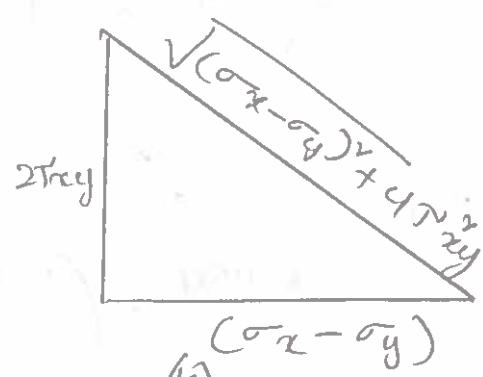
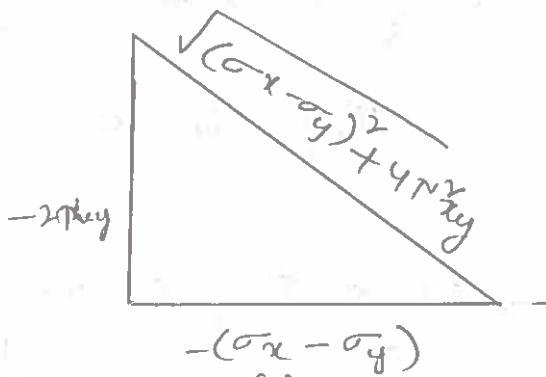
Now resolving the forces tangential to AB.

$$\begin{aligned} P_t &= P_1 \cos\theta + P_2 \sin\theta - P_3 \sin\theta - P_4 \cos\theta \\ &= \sigma_x AC \cos\theta + \tau_{xy} AC \sin\theta - \sigma_y BC \sin\theta - \tau_{xy} BC \cos\theta \end{aligned}$$

Normal stress (across the inclined section AB)

$$\begin{aligned} \sigma_n &= \frac{P_n}{AB} \\ &= \frac{\sigma_x AC \sin\theta - \tau_{xy} AC \cos\theta + \sigma_y BC \cos\theta - \tau_{xy} BC \sin\theta}{AB} \\ &= \frac{\sigma_x AC \sin\theta}{AB} - \frac{\tau_{xy} AC \cos\theta}{AB} + \frac{\sigma_y BC \cos\theta}{AB} - \frac{\tau_{xy} BC \sin\theta}{AB} \\ &= \frac{\sigma_x AC \sin\theta}{AC \sin\theta} - \frac{\tau_{xy} AC \cos\theta}{AC \sin\theta} + \frac{\sigma_y BC \cos\theta}{BC \cos\theta} - \frac{\tau_{xy} BC \sin\theta}{BC \cos\theta} \\ &= \sigma_x \sin^2\theta - \tau_{xy} \sin\theta \cos\theta + \sigma_y \cos^2\theta - \tau_{xy} \sin\theta \cos\theta \end{aligned}$$

$$\tan 2\theta_p = \frac{2\tau_{xy}}{(\sigma_x - \sigma_y)}$$



thus we find that there are two principal planes, at right angles to each other, their inclinations with x-x axis being θ_{p_1} and θ_{p_2}

Now for case 1 $\sin 2\theta_{p_1} = \frac{-2\tau_{xy}}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}}$

$\& \cos 2\theta_{p_1} = \frac{-(\sigma_x - \sigma_y)}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}}$

Similarly for case 2.

$$\sin 2\theta_{p_2} = \frac{2\tau_{xy}}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}} \& \cos 2\theta_{p_2}$$

$$\cos 2\theta_{p_2} = \frac{(\sigma_x - \sigma_y)}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}}$$

$$= \frac{\sigma_x}{2} (1 - \cos 2\theta) + \frac{\sigma_y}{2} (1 + \cos 2\theta) - T_{xy} \sin \theta \cos \theta$$

$$= \frac{\sigma_x}{2} - \frac{\sigma_x}{2} \cos 2\theta + \frac{\sigma_y}{2} + \frac{\sigma_y}{2} \cos 2\theta + \frac{\sigma_y}{2} + \frac{\sigma_y}{2} \cos 2\theta - T_{xy} \sin \theta.$$

$$\sigma_n = \frac{\sigma_x + \sigma_y}{2} - \left(\frac{\sigma_x - \sigma_y}{2} \right) \cos 2\theta - T_{xy} \sin 2\theta \quad - ⑤$$

Shear stress or tangential stress across inclined section AB)

$$\tau = \frac{P_t}{AB} = \frac{\sigma_x AC \cos \theta + T_{xy} AC \sin \theta - \sigma_y BC \sin \theta - T_{xy} BC \cos \theta}{AB}$$

$$= \frac{\sigma_x AC \cdot \cos \theta}{AB} + \frac{T_{xy} AC \sin \theta}{AB} - \frac{\sigma_y BC \sin \theta}{AB} - \frac{T_{xy} BC \cos \theta}{AB}$$

$$= \sigma_x \sin \theta \cos \theta + T_{xy} \sin^2 \theta - \sigma_y \sin \theta \cos \theta - T_{xy} \cos^2 \theta$$

$$= (\sigma_x - \sigma_y) \sin \theta \cos \theta + \frac{T_{xy}}{2} (1 - \cos 2\theta) - \frac{T_{xy}}{2} (1 + \cos 2\theta)$$

$$\tau = \left[\frac{\sigma_x - \sigma_y}{2} \right] \sin 2\theta - T_{xy} \cos 2\theta \quad - ⑥$$

The value of the angle for which the shear stress is zero

$$\therefore \frac{\sigma_x - \sigma_y}{2} \sin 2\theta_p - T_{xy} \cos 2\theta_p = 0$$

$$\frac{\sigma_x - \sigma_y}{2} \sin 2\theta_p = T_{xy} \cos 2\theta_p$$

(7)

∴ maximum principal stress,

$$\sigma_{P_1} = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2} \cos 2\theta - T_{xy} \sin 2\theta$$

$$= \frac{\sigma_x + \sigma_y}{2} - \left[\frac{\sigma_x - \sigma_y}{2} \times \frac{-(\sigma_x - \sigma_y)}{\sqrt{(\sigma_x - \sigma_y)^2 + 4T_{xy}^2}} \right] - \left[T_{xy} \frac{-2T_{xy}}{\sqrt{(\sigma_x - \sigma_y)^2 + 4T_{xy}^2}} \right]$$

$$= \frac{\sigma_x + \sigma_y}{2} + \frac{(\sigma_x - \sigma_y)^2 + 4T_{xy}^2}{2\sqrt{(\sigma_x - \sigma_y)^2 + 4T_{xy}^2}}$$

$$= \frac{\sigma_x + \sigma_y}{2} + \frac{\sqrt{(\sigma_x - \sigma_y)^2 + 4T_{xy}^2}}{2}$$

$$\boxed{\sigma_{P_1} = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left[\frac{\sigma_x - \sigma_y}{2} \right]^2 + T_{xy}^2}}$$

Minimum principle stress:

$$\sigma_{P_2} = \frac{\sigma_x + \sigma_y}{2} - \frac{(\sigma_x - \sigma_y)}{2} \cos 2\theta - T_{xy} \sin 2\theta$$

$$= \frac{\sigma_x + \sigma_y}{2} - \left[\frac{\sigma_x - \sigma_y}{2} \times \frac{(\sigma_x - \sigma_y)}{\sqrt{(\sigma_x - \sigma_y)^2 + 4T_{xy}^2}} \right] - \left[T_{xy} \frac{2T_{xy}}{\sqrt{(\sigma_x - \sigma_y)^2 + 4T_{xy}^2}} \right]$$

$$= \frac{\sigma_x + \sigma_y}{2} - \frac{(\sigma_x - \sigma_y)^2 + 4T_{xy}^2}{2\sqrt{(\sigma_x - \sigma_y)^2 + 4T_{xy}^2}}$$

$$= \frac{\sigma_x + \sigma_y}{2} - \frac{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}}{2}$$

$$\sigma_{P_2} = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left[\frac{\sigma_x - \sigma_y}{2} \right]^2 + \tau_{xy}^2}$$

Graphical Method for the Stresses On An oblique Section of A Body :-

Graphical method is used to determination of normal, shear and resultant stress across a section. This is done by drawing a mohr's circle stresses.

The construction of mohr's circle of stresses as well as determination of normal, shear & resultant stresses is very easier than the analytical method. we shall draw the mohr's circle of stresses for the following cases.

1. A body subjected to a direct stress in one plane.
2. A body subjected to direct stresses in two mutually perpendicular directions.
3. A body subjected to a simple shear stress.

- (B) 4. A body subjected to a direct stress in one plane accompanied by a simple shear stress.
5. A body subjected to direct stresses in two mutually perpendicular directions accompanied by a simple shear stress.

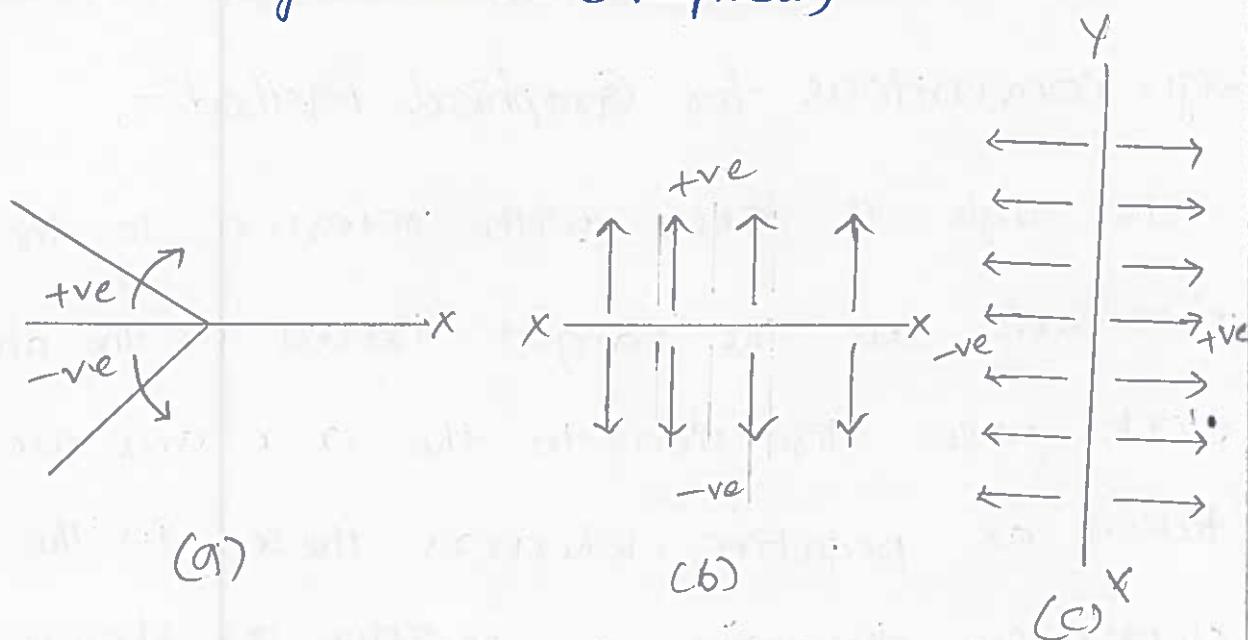
Sign conventions for Graphical Method:-

The angle θ is taken with reference to the $x-x$ axis all the angles traced in the anti clockwise direction to the $x-x$ axis are taken as negative, whereas those in the clockwise direction as positive as shown in fig in the clock wise direction as positive as shown in fig. The value of angle θ , until told unless mentioned is taken as positive and drawn clock wise.

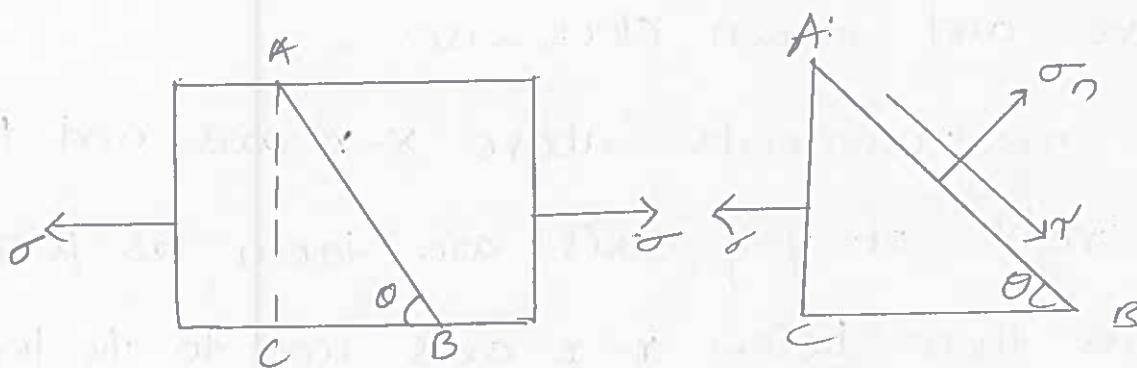
2. The measurements above $x-x$ axis and to the right of $y-y$ axis are taken as positive whereas those below $x-x$ axis and to the left of $y-y$ axis as negative as shown in fig (b) & (c)
3. sometimes there is a slight variation in the results obtained by analytical method and graphical method. The values obtained by graphical method are taken to be correct if -

they agree upto the first decimal correct if they agree upto the first decimal point with values obtained by analytical method.

e.g 8.66 (Analytical) = 8.7 (Graphical), similarly
 4.32 (Analytical) = 4.3 (Graphical)



Mohr's circle for stresses on an oblique section of body subjected to a direct stress in one plane:-



Consider (a) a rectangular body of uniform cross-sectional area and unit thickness subjected to a direct tensile stress along x-x axis as shown in fig (a) & (b).

(14)

Now let us consider an oblique section AB inclined with x-x axis

let σ = tensile stress (x-x)

θ = angle of inclination

→ consider the equilibrium of the wedge ABC.

Now draw the Mohr's circle of stresses as shown in fig and as discussed below

1. first of all, take some suitable point O and through it draw a horizontal line XOX.
2. cut off OJ equal to the tensile stress (σ) to some suitable scale and towards right (because σ is tensile). Bisect OJ at C. Now the point O represents the stress system on plane BC and the point J represents the stress system on plane AC.
3. Now with 'C' as centre and radius equal to 'CO' and 'CJ' draw a circle. It is known as mohr's circle for stresses.
4. Now through 'C' draw a line CP making an angle of 2θ with CO in the clockwise direction meeting the circle at P. the point R represents the section AB.

through P, draw PQ perpendicular to OX.
Join OP. Now OQ, QP & OP will give the normal stresses, shear stress and resultant stresses respectively to the scale. And the angle POJ is called the angle of obliquity (θ)

Proof:-

from the geometry of the mohr's circle of stress, we find that

$$OC = CQ = CP = \sigma/2 \quad (\because \text{Radius of the circle})$$

∴ Normal stress

$$\sigma_n = OQ = OC - QC = \left(\frac{\sigma}{2}\right) - \left(\frac{\sigma}{2}\right) \cos 2\theta$$

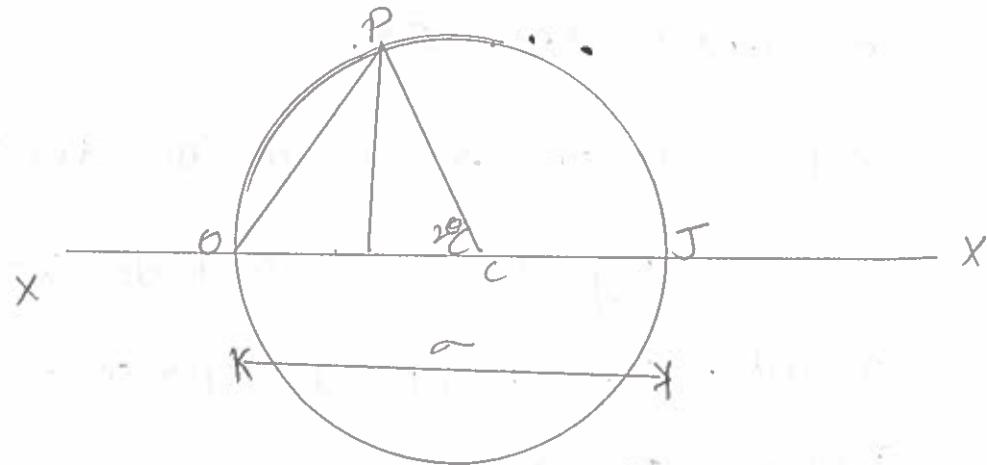
and Shear stress

$$\tau = QP = CP \sin 2\theta = \frac{\sigma}{2} \sin 2\theta$$

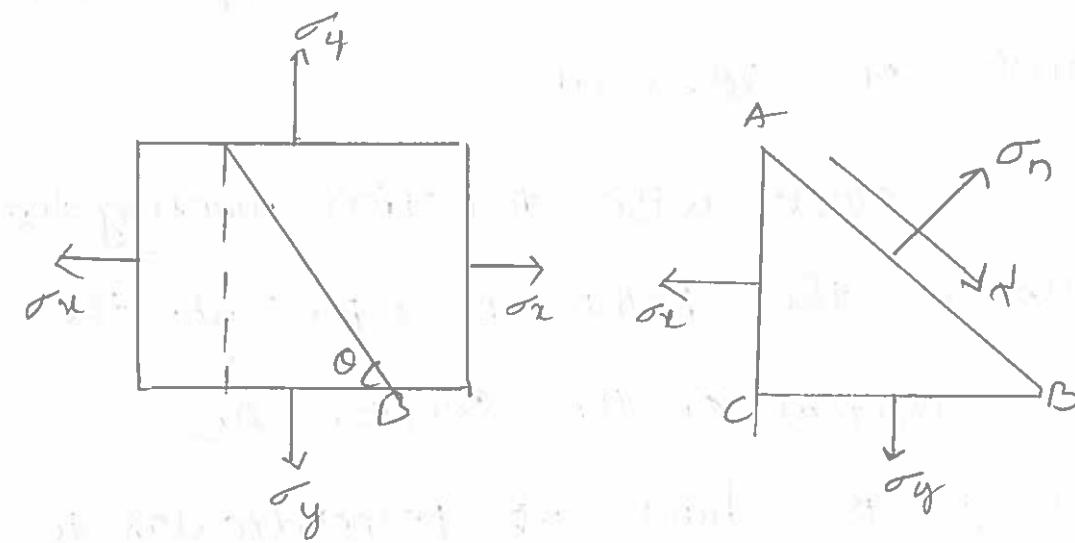
we also find that maximum shear stress will be equal to the radius of the mohr's circle of stresses i.e. $\sigma/2$. It will happen when θ is equal to 90° or 270° i.e. θ is equal to 45° is 135° .

However when $\theta=45^\circ$ then the shear stress is equal to $\sigma/2$ and when $\theta=135^\circ$ then the shear stress is equal to $-\sigma/2$.

(5)



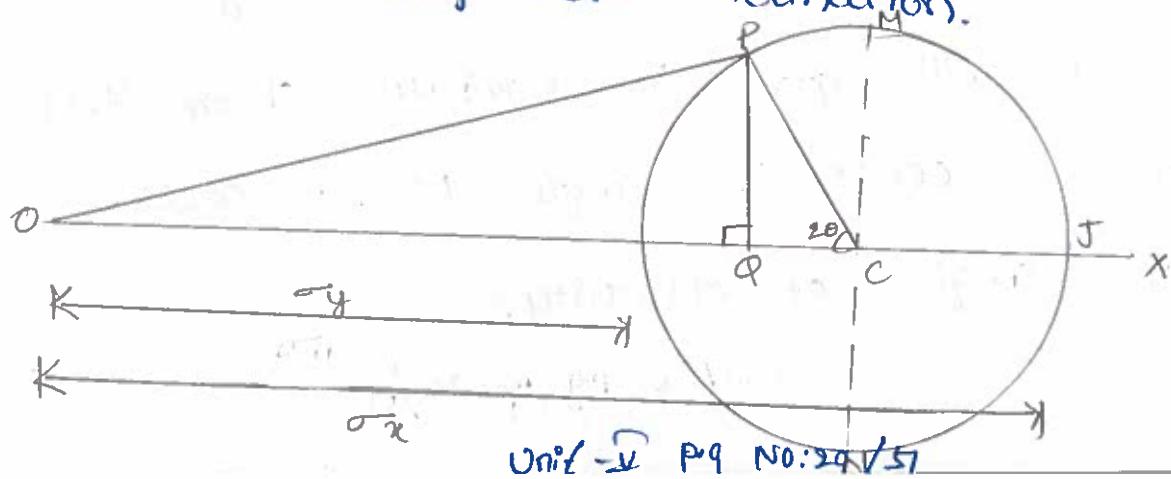
Mohr's Circle for Stresses on An Oblique Section of A Body subjected To Direct Stresses In Two Mutually perpendicular Directions



Let σ_x = Tensile Stress in x-x direction
(Major)

σ_y = Tensile Stress in y-y direction
(minor)

θ = angle of inclination.



1. first of all take some suitable point O and draw a horizontal line OX.
2. cut off OJ and OK equal to the tensile stresses σ_x and σ_y to some suitable scale towards right. the point J represent the stress system on plane BC. Bisect JK at C.
3. Now with 'C' as centre and radius equal to CJ or CK draw a circle. It is known as mohr's circle of stresses.
4. Now through C, draw a line CP making an angle of 2θ with CK in clockwise direction meeting the circle at the point P represent the stress system on the section AB.
5. through P, draw PQ perpendicular to the line OX, join OP.
6. Now OQ, QP & OP will give the normal stress, shear stress and resultant stress respectively to the scale. similarly CM or CN will give the maximum shear stress to the scale the angle POC. is called the angle of obliquity.

(6) Proof from the geometry of the Mohr's circle of stress we find that

$$KC = CJ = CP = \frac{\sigma_x - \sigma_y}{2}$$

$$OC = OK + KC = \sigma_y + \frac{\sigma_x - \sigma_y}{2} = \frac{2\sigma_y + \sigma_x - \sigma_y}{2} = \frac{\sigma_x + \sigma_y}{2}$$

$$\therefore \text{Normal stress } \sigma_n = OQ = OC - CQ$$

$$= \frac{\sigma_x + \sigma_y}{2} - CD \cos 2\theta$$

$$= \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2} \cos 2\theta$$

$$\text{and shear stress } \Rightarrow \tau = OP = CP \sin 2\theta$$

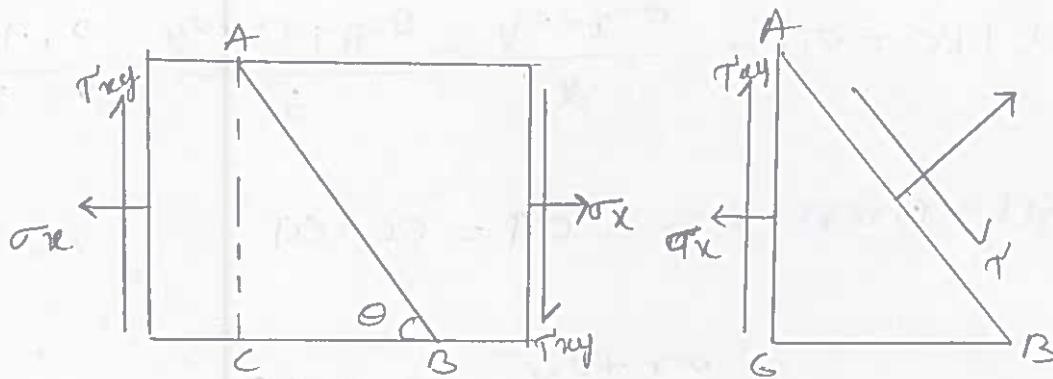
$$= \frac{\sigma_x + \sigma_y}{2} \sin 2\theta.$$

we also find that the maximum shear stress will be equal to the radius of the mohr's circle of stresses i.e $\frac{\sigma_x - \sigma_y}{2}$. It will happen when 2θ is equal to 90° or 270° i.e when θ is equal to 45° or 135° .

However when $\theta = 45^\circ$, then the shear stress is equal to $\left[\frac{\sigma_x - \sigma_y}{2} \right]$

when $\theta = 135^\circ$, then the shear stress will be equal to $-\frac{(\sigma_x - \sigma_y)}{2}$ or $\left(\frac{\sigma_y - \sigma_x}{2} \right)$.

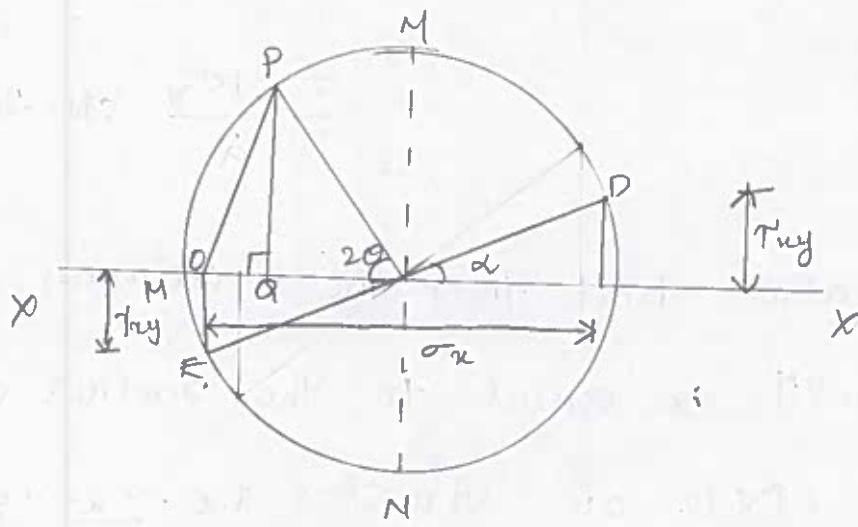
Mohr's circle for stresses on an oblique section of body subjected to a direct stresses in one plane Accompanied by
 a) simple shear stress -



let σ_x = ^(a) tensile stress in $x-x$ direction ^(b)

τ_{xy} = positive (i.e. clockwise)

θ = angle of inclination



- i) Take some suitable point 'O' and through it draw a horizontal line xox' .
- ii) Cut off OT equal to the tensile stress σ_x .
- iii) Now erect a perpendicular at T above the line $x-x'$. and cut off TD equal to the shear stress τ_{xy} to the scale. similarly

(7) event a perpendicular below the line $\sigma_x - \sigma_y$
and cut off σ_E equal to the shear stress
try to the scale. join DE and bisect it at C .

4. Now with C as centre and radius equal
to CD or CE draw a circle. It is known
as mohr's circle of stresses.

5. Now through C , draw a line CP making an
angle 2θ with CE in clock wise direction
meeting the circle at P .

6. Through P , draw PQ perpendicular to the
line OX join OP .

Now OQ , QP & OP will give the normal, shear
& resultant stresses to the scale.

Proof: from the geometry of the mohr's circle
of stresses

$$OC = \frac{\sigma_x}{2}$$

and radius of the circle

$$R = EC = CD = CP = \sqrt{\left(\frac{\sigma_x}{2}\right)^2 + r_{xy}^2}$$

Now in the right angled triangle DCP

$$\sin \alpha = \frac{DC}{CP} = \frac{r_{xy}}{R} \text{ and } \cos \alpha = \frac{JC}{CD} = \frac{\sigma_x}{2R}$$

$$\therefore = \frac{\sigma_x}{2R}$$

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and similarly in right angled triangle CPQ,

$$\angle PCQ = (\alpha - \theta)$$

$$CQ = CP \cos(\alpha - \theta) = R[\cos(\alpha - \theta)]$$

$$= R[\cos\alpha \cdot \cos\theta + \sin\alpha \sin\theta]$$

$$= R \cos\alpha \cdot \cos\theta + R \sin\alpha \sin\theta \quad \begin{bmatrix} \cos(a-b) = \\ \cos a \cos b + \sin a \sin b \end{bmatrix}$$

$$= R \frac{\sigma_x}{2R} \cos\theta + R \frac{T_{xy}}{R} \sin\theta$$

we know that normal stress across the section AB,

$$\sigma_x = OQ = OC - CQ = \frac{\sigma_x}{2} - \left[\frac{\sigma_x}{2} \cos\theta + T_{xy} \sin\theta \right]$$

$$= \frac{\sigma_x}{2} - \frac{\sigma_x}{2} \cos\theta - T_{xy} \sin\theta$$

and shear stress, $T' = OP = CP \sin(\alpha - \theta) = R \sin(\alpha - \theta)$

$$= R[\cos\alpha \cdot \sin\theta - \sin\alpha \cos\theta]$$

$$= R \cos\alpha \cdot \sin\theta - R \sin\alpha \cos\theta$$

$$= R \frac{\sigma_x}{2R} \sin\theta - R \frac{T_{xy}}{R} \cos\theta$$

$$= \frac{\sigma_x}{2} \sin\theta - T_{xy} \cos\theta$$

we also know that maximum stress.

$$\sigma_{max} = OG = OC + CG = \frac{\sigma_x}{2} + \sqrt{\left(\frac{\sigma_x}{2}\right)^2 + T_{xy}^2}$$

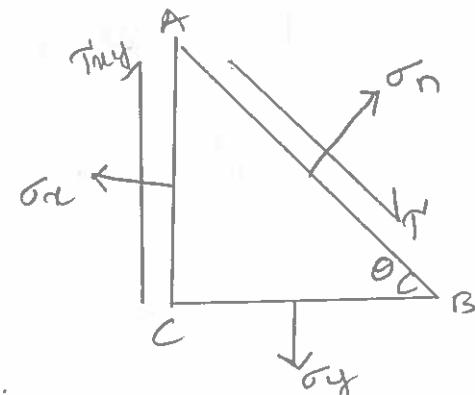
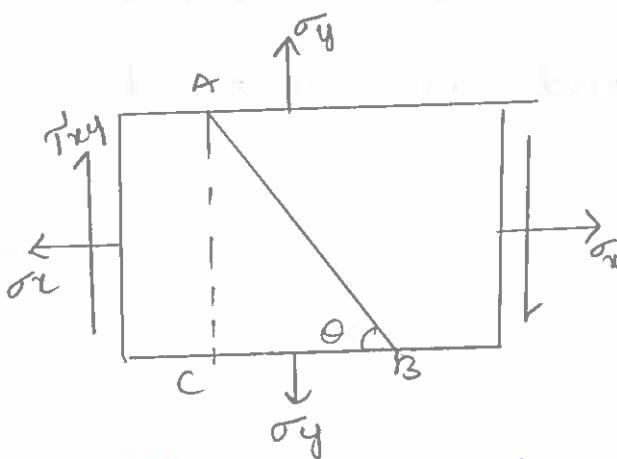
(18) And minimum stress

$$\sigma_{\min} = OM = OC - CH = \frac{\sigma_x}{2} - \sqrt{\left(\frac{\sigma_x}{2}\right)^2 + \tau_{xy}^2}$$

we also find that the maximum shear stress will be equal to the radius of the mohr's circle of stresses i.e. $\sqrt{\left(\frac{\sigma_x}{2}\right)^2 + \tau_{xy}^2}$. It will happen when $(2\theta - \alpha)$ is equal to 90° or 270° .

However when $(2\theta - \alpha)$ is equal to 90° then the shear stress is equal to $+\sqrt{\left(\frac{\sigma_x}{2}\right)^2 + \tau_{xy}^2}$ and when $(2\theta - \alpha) = 270^\circ$ then the shear stress is equal to $-\sqrt{\left(\frac{\sigma_x}{2}\right)^2 + \tau_{xy}^2}$.

Mohr's Circle for stresses on an oblique section of a body subjected to direct stresses in two mutually perpendicular directions accompanied by a shear stress:



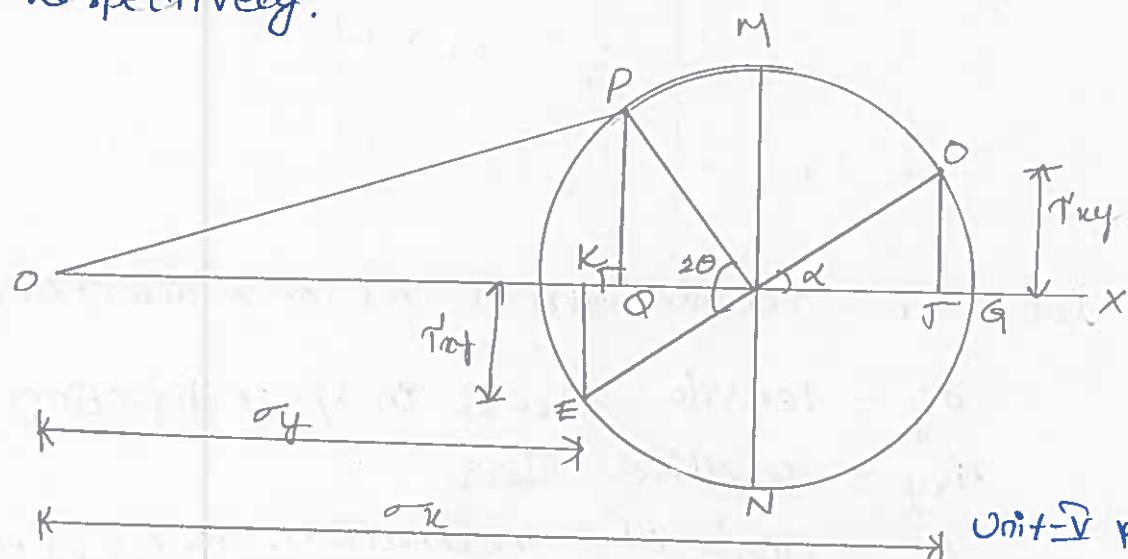
let σ_x = tensile stress in $x-x$ direction

σ_y = tensile stress in $y-y$ direction

τ_{xy} = positive shear

θ = Angle of inclination. Unit Pg No: 35/5

1. Take some suitable point 'O' draw horizontal line OX .
 2. cut off OJ and OK equal to the tensile stresses σ_x & σ_y respectively.
 3. Let now meet a perpendicular J above the line $X-X$ and cut off JD equal to the shear stress try to the scale.
 4. Now with 'C' as centre & radius equal to CD & CJ draw a circle it is known as mohr's circle.
 5. Now through C draw a line CP making an angle 2θ with CE in clockwise directions meeting the circle at P the point P represents the stress system on section AB.
 6. Through P, draw PQ perpendicular to the line OX join OP .
 7. Now OQ , QP & OP will give the normal stress, shear stress and resultant stress respectively.



(12)

Proof: From the geometry of the Mohr's circle of stress we find that

$$OC = \frac{\sigma_x + \sigma_y}{2}$$

and radius of the circle

$$R = EC = CD = CP = \sqrt{\left[\frac{\sigma_x - \sigma_y}{2}\right]^2 + \tau_{xy}^2}$$

Now in the right angled triangle OCJ

$$\sin \alpha = \frac{JD}{OC} = \frac{\tau_{xy}}{R} \text{ & } \cos \alpha = \frac{CJ}{OC} = \frac{\sigma_x - \sigma_y}{2R}$$

similarly in right angle triangle CPA.

$$\angle PCQ = (\angle \theta - \alpha)$$

$$CP = CP \cos(\angle \theta - \alpha)$$

$$= R [\cos(\angle \theta - \alpha)]$$

$$= R [\cos \alpha \cdot \cos \angle \theta + \sin \alpha \cdot \sin \angle \theta]$$

$$= R \cos \alpha \cos \angle \theta + R \sin \alpha \sin \angle \theta$$

$$= R \frac{\sigma_x - \sigma_y}{2R} \cos \angle \theta + R \cdot \frac{\tau_{xy}}{R} \sin \angle \theta$$

$$= \frac{\sigma_x - \sigma_y}{2} \cos \angle \theta + \tau_{xy} \sin \angle \theta.$$

Normal stresses across the inclined section AB

$$\sigma_n = OQ = OC - CQ$$

$$\sigma_n = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2} \cos 2\theta - T_{xy} \sin 2\theta$$

Shear stress (across the inclined section AB)

$$\tau = QP = C_P [\sin(2\theta - \alpha)]$$

$$= R [\sin(2\theta - \alpha)]$$

$$= R \cos \alpha \cdot \sin 2\theta - R \sin \alpha \cos 2\theta$$

$$= R \cdot \frac{\sigma_x - \sigma_y}{2R} \sin 2\theta - R \frac{T_{xy}}{R} \cos 2\theta$$

$$\tau = \frac{\sigma_x - \sigma_y}{2} \sin 2\theta - T_{xy} \cos 2\theta.$$

Maximum principle stress:

$$\sigma_{max} = OG = OC + CA = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left[\frac{\sigma_x - \sigma_y}{2}\right]^2 + T_{xy}^2}$$

Minimum principle stresses:

$$\sigma_{min} = OM = OC - CM$$

$$= \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left[\frac{\sigma_x - \sigma_y}{2}\right]^2 + T_{xy}^2}$$

we also find the maximum shear stresses will be equal to the radius of the Mohr's circle

of stresses i.e. $\sqrt{\left[\frac{\sigma_x - \sigma_y}{2}\right]^2 + T_{xy}^2}$ it will happen

when $(2\theta - \alpha)$ is equal to 90° or 270° .

PART - II

Theories of failure:

When some external load is applied on a body, the stresses and strains are produced in the body. The stresses are directly proportional to the strain within the elastic limit.

Certain theories have advanced to explain the cause of failure. According to the important theories the failure takes place when a certain limiting value is reached by one of the following:

1. The maximum principle stress (Rankine) 
 2. The maximum principle strain (St. Venant) 
 3. The maximum shear stress (Guest & Tresca) 
 4. The maximum strain energy (Hauges) 
 5. The maximum shear strain energy (von-mises)
- Hence 

In all the above cases

$\sigma_1, \sigma_2, \sigma_3$ = principle stresses in my complex system

σ^* = tensile or compressive stress at the elastic limit.

Brittle: compression > shear > tension

ductile: compression > tension > shear.

i) maximum principle stress theory:-

According to this theory, the failure of a material will occur when the maximum principal tensile stress (σ_1) in the complex system reaches the value in simple tension or the minimum principle stress reaches the value of the maximum stress at the elastic limit in simple compression.

Let in a complex three dimensional stress system $\sigma_1, \sigma_2, \sigma_3$ = principle stresses at a point in three perpendicular directions. The stresses $\sigma_1 \& \sigma_2$ are tensile σ_3 is compressive

σ_t^* = tensile stress at elastic limit in simple tensile stress

σ_c^* = compressive stress at elastic limit in simple compression

Then according to this theory, the failure will take place if

$$\sigma_1 \geq \sigma_t^* \text{ in simple tension}$$

$$|\sigma_3| \geq \sigma_c^* \text{ in simple compression.}$$

(21)

where $|\sigma_3|$ represents the absolute value of σ_3 .

This is the simplest and oldest theory of failure and is known as Rankine's theory. If the maximum principal stress (σ_1) is the design criterion, then maximum principle stress must not exceed the permissible stress (σ_t) for the given material.

$$\text{Hence } \sigma_1 = \sigma_t$$

where σ_t = permissible stress and is given

by

$$\sigma_t = \frac{\sigma_{t*}}{\text{factor of safety}}$$

2. Maximum principal strain theory:

This theory is due to Saint Venant.

According to this theory the failure will occur in a material when the maximum principle strain reaches the strain due to yield stress in simple tension or when the minimum principle strain.

principal strain in the direction of principle stress σ_1 is

$$e_1 = \frac{\sigma_1}{E} - \frac{M\sigma_2}{E} - \frac{M\sigma_3}{E}$$

$$= \frac{1}{E} [\sigma_1 - N(\sigma_2 + \sigma_3)]$$

principle stress in the direction of principle stress σ_3 is

$$\epsilon_3 = \frac{1}{E} [\sigma_3 - N(\sigma_1 + \sigma_2)]$$

strain due to yield stress in simple tension

$$= \frac{1}{E} \times \text{yield stress in tension}$$

$$= \frac{1}{E} \times \sigma_t^*$$

and strain due to yield stress in simple

$$\text{compression} = \frac{1}{E} \times \sigma_c^*$$

where yield stress is the maximum stress at elastic limit.

According to this theory the failure of the material will take place when.

$$\epsilon_1 \geq \frac{\sigma_t^*}{E}$$

(or)

$$|\epsilon_3| \geq \frac{\sigma_c^*}{E}$$

Substituting the values of ϵ_1 and ϵ_3 , we get the conditions of failure as:

$$(i) \frac{1}{E} [\sigma_1 - N(\sigma_2 + \sigma_3)] \geq \frac{1}{E} \times \sigma_t^* \quad \text{--- } ①$$

$$\text{(or)} \quad \sigma_1 - N(\sigma_2 + \sigma_3) \geq \sigma_t^*$$

$$(ii) \left| \frac{1}{E} [\sigma_3 - N(\sigma_1 + \sigma_2)] \right| \geq \frac{1}{E} \times \sigma_c^*$$

$$\left| [\sigma_3 - N(\sigma_1 + \sigma_2)] \right| \geq \sigma_c^* \quad \text{--- } ②$$

(v) for actual design (i.e. where some quantity is to be calculated) instead of σ_t^* or σ_c^* . the permissible stresses (σ_t or σ_c) in simple tension or compression should be used when

$$\sigma_t = \frac{\sigma_t^*}{\text{factor of safety}}$$

$$\sigma_c = \frac{\sigma_c^*}{\text{factor of safety}}$$

and

hence for design purpose the equations

③ & ④ beam

$$\sigma_1 - \mu(\sigma_2 + \sigma_3) = \sigma_t \quad \text{--- (iii)}$$

and

$$[\sigma_3 - \mu(\sigma_1 + \sigma_2)] = \sigma_c \quad \text{--- (iv)}$$

equations ③ & ④ should be used for design purposes. They should not be used for determining the failure of the material.

Maximum shear stress theory:

This theory is due to Guest and theory and therefore known as Guest's theory. According to this theory the failure of a material will occur when the maximum shear stress

In a material reaches the value of maximum shear stress in simple tension at the elastic limit.

If $\sigma_1, \sigma_2, \sigma_3$ are principle stresses at a point in a material for which σ_t^* is the principal stresses in simple tension at elastic limit, then

max shear stress in the material = Half of difference of maximum and minimum principle stresses.

$$= \frac{1}{2} [\sigma_1 - \sigma_3]$$

In case of simple tension, at the elastic limit the principal stresses are $\sigma_t^*, 0, 0$.

\therefore Max shear stress in simple tension at elastic limit.

= half of difference of maximum & minimum principle stress

$$= \frac{1}{2} [\sigma_t^* - 0] = \frac{\sigma_t^*}{2}$$

\therefore for the failure of material

$$\frac{1}{2} (\sigma_1 - \sigma_3) \geq \frac{1}{2} \sigma_t^* \text{ (or)} (\sigma_1 - \sigma_3) \geq \sigma_t^*$$

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for actual design (i.e. when some quantity is to be calculated) instead of σ_{F^+} . The allowable stress (σ_t) in simple tension should be considered.

$$\text{where } \sigma_t = \frac{\sigma_{F^+}}{\text{Safety factor}}$$

Hence for design, the following equations should be used $(\sigma_1 - \sigma_3) = \sigma_t$

The above equation is to be used for design purpose only. It should ^{not} be used for determining the failure of the material due to maximum shear theory.

Maximum Strain Energy Theory:

This theory is due to Hooke and is known as Hooke theory. According to this theory the failure of a material occurs when the total strain energy per unit volume of the material at the elastic limit in simple tension

We have stated that the strain energy in a body is equal to work

done by the load (P) in straining the material and is equal to

$$\frac{1}{2} \times P \times \delta L$$

$V = \text{strain energy}$

(\therefore load $\frac{1}{2}$ gradually increased from 0 to P)

$$= \frac{1}{2} \times P \times \delta L$$

$$= \frac{1}{2} \times (\sigma \times A) \times (e) \times L$$

$$[\because \sigma = P/A]$$

$$= \frac{1}{2} \times \sigma \times e \times A \times L$$

$$e = \frac{\delta L}{L}$$

$$= \frac{1}{2} \times \text{stress} \times \text{strain} \times \text{volume}$$

\therefore strain energy per unit volume

$$= \frac{1}{2} \times \text{stress} \times \text{strain}$$

$$= \frac{1}{2} \times \sigma \times e$$

for a three dimensional stress system the principal stresses acting at a point are σ_1 , σ_2 and σ_3 . The corresponding strains are e_1 , e_2 & e_3 . where $e_1 = \text{principle strain in the direction of } \sigma_1$

$$\text{Now } e_1 = \frac{\sigma_1}{E} - \frac{1}{E} (\sigma_2 + \sigma_3)$$

$$\text{similarly } e_2 = \frac{\sigma_2}{E} - \frac{1}{E} (\sigma_3 + \sigma_1)$$

$$e_3 = \frac{\sigma_3}{E} - \frac{1}{E} (\sigma_1 + \sigma_2).$$

\therefore Total strain energy per unit volume in three dimensional system.

$$\begin{aligned}
 U &= \frac{1}{2} \times \sigma_1 \times e_1 + \frac{1}{2} \times \sigma_2 \times e_2 + \frac{1}{2} \times \sigma_3 \times e_3 \\
 &= \frac{1}{2} \sigma_1 \left[\frac{\sigma_1}{E} - \frac{M}{E} (\sigma_2 + \sigma_3) \right] + \frac{1}{2} \times \sigma_2 \times \left[\frac{\sigma_2}{E} - \frac{M}{E} (\sigma_3 + \sigma_1) \right] \\
 &\quad + \frac{1}{2} \times \sigma_3 \left[\frac{\sigma_3}{E} - \frac{M}{E} (\sigma_1 + \sigma_2) \right] \\
 &= \frac{1}{2E} [\sigma_1^r + \sigma_2^r + \sigma_3^r - 2M(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)]
 \end{aligned}$$

The strain energy per unit volume corresponding to stress.

Elastic limit in simple tension

$$\begin{aligned}
 &= \frac{1}{2} \times \sigma_t^* \times e_t^* \quad \text{[where } e_t^* = \text{strain due to } \sigma_t^*] \\
 &= \frac{1}{2} \times \sigma_t^* \times \frac{\sigma_t^*}{E} \\
 &= \frac{1}{2E} (\sigma_t^*)^2
 \end{aligned}$$

for the failure of the material

$$\frac{1}{2E} [\sigma_1^r + \sigma_2^r + \sigma_3^r - 2M(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)] \geq \frac{1}{2E} \times (\sigma_t^*)^2$$

for a two-dimensional stress system $\sigma_3 = 0$;

Hence above equation becomes as

$$\sigma_1^r + \sigma_2^r - 2M(\sigma_1\sigma_2) \geq (\sigma_t^*)^2$$

for a two-dimensional stress system $\sigma_3 = 0$;
the above equation becomes as

$$\sigma_1^r + \sigma_2^r - 2M(\sigma_1, \sigma_2) \geq (\sigma_t^*)^2$$

for actual design (i.e. when some quantity
is to be calculated instead of σ_t^* , the
allowable stress (σ_t) in simple tension
should be considered where.

$$\sigma_t = \frac{\sigma_t^*}{\text{factor of safety}}$$

Hence for design the following equation
should be used.

$$\sigma_1^r + \sigma_2^r - 2M(\sigma_1, \sigma_2) = \sigma_t^r$$

The above equation is used for design
purpose only. It is not used for
determining the failure of material due
to maximum strain energy theory.

Maximum shear strain energy theory :-

(Non Mises theory):

This theory is due to Mises and
Henley and is known as Mises-Henley
theory. This theory is also called the energy
of distortion theory. According to this theory
the failure of a material occurs when

(26) when the total shear strain energy per unit volume in the stressed material reaches a value equal to the shear strain energy per unit volume at the elastic limit in the simple tension test.

The total shear strain energy (or energy of distortion) per unit volume due to principal stresses $\sigma_1, \sigma_2 \neq \sigma_3$ in a stressed material is given as

$$= \frac{1}{12c} [(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]$$

The simple tension test is a uniaxial stress system which means the principal stresses are $\sigma_1, 0, 0$.

at the elastic limit the tensile stress in simple test is σ_E^*

Hence at the elastic limit in simple tension test, the principal stresses are $\sigma_E^*, 0, 0$.

The shear strain energy per unit volume at the elastic limit in simple tension will be

$$= \frac{1}{12c} [(\sigma_E^* - 0)^2 + (0 - 0)^2 + (0 - \sigma_E^*)^2]$$

$$= \frac{1}{12c} [2 \times \sigma_E^{*2}] \quad [\text{Here } \sigma_1 = \sigma_E^*, \sigma_2 = 0; \\ \sigma_3 = 0] \\ \text{Unit } \sqrt{\text{Pg Nm}^2/\text{m}^3}$$

for the failure of the material

$$\frac{1}{12c} [(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2] \geq \frac{1}{12c} [2 \times (\sigma_E^*)^2]$$

(or)

$$[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2] \geq 2(\sigma_E^*)^2.$$

for actual design, instead of σ_E^* the allowable stress (σ_t) in simple tension should be used where.

$$\sigma_t = \frac{\sigma_t^*}{\text{safety factor}}$$

Hence for design purpose, the following equation should be used.

$$(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 = 2 \times \sigma_t^2$$

The above equation should be used for design purpose only. It should not be used for determining the failure of the two-dimensional stress system, $\sigma_3 = 0$.

Hence above equation becomes as

⑥

$$(\sigma_1 - \bar{\sigma}_2)^2 + (\bar{\sigma}_2 - 0)^2 + (0 - \sigma_1)^2 = 2 \times \sigma_t^2$$

$$\sigma_1^2 + \bar{\sigma}_2^2 - 2\sigma_1\bar{\sigma}_2 + \bar{\sigma}_2^2 + \sigma_1^2 = 2\bar{\sigma}_t^2$$

$$2(\sigma_1^2 + \bar{\sigma}_2^2 - \sigma_1\bar{\sigma}_2) = 2\bar{\sigma}_t^2$$

$$\boxed{\sigma_1^2 + \bar{\sigma}_2^2 - \sigma_1\bar{\sigma}_2 = \sigma_t^2}$$

Unit \rightarrow P.g No: s^{-1}/s^{-1}

